

## SOME RESULTS OF THE MARIÑO-VAFA FORMULA

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**ABSTRACT.** In this paper we derive some new Hodge integral identities by taking the limits of Mariño-Vafa formula. These identities include the formula of  $\lambda_1 \lambda_g$ -integral on  $\overline{\mathcal{M}}_{g,1}$ , the vanishing result of  $\lambda_g \text{ch}_{2l}(\mathbb{E})$ -integral on  $\overline{\mathcal{M}}_{g,1}$  for  $1 \leq l \leq g-3$ . Using the differential equation of Hodge integrals, we give a recursion formula of  $\lambda_{g-1}$ -integrals. Finally, we give two simple proofs of  $\lambda_g$  conjecture and some examples of low genus integral.

## 1. Introduction

Based on string duality, Mariño and Vafa [10] conjectured a closed formula on certain Hodge integrals in terms of representations of symmetric groups. Recently, C.C. Liu, K. Liu and J. Zhou [6] proved this formula and derived some consequences from it [7]. In this paper we follow their method to derive some new Hodge integral identities. One of the main results of this paper is the following identity: if  $1 \leq m \leq 2g-3$ , then

$$\begin{aligned}
 (1) \quad & -(2g-2-m)!(-1)^{2g-3-m} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2g-2-m}(\mathbb{E}) \psi_1^m \\
 &= b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m}}{2g-1-k} B_{2g-1-m} \\
 &+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, m-1)} \frac{(-1)^{2g_2-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2g-1-m}}{2g-1-k} B_{2g-1-m}.
 \end{aligned}$$

As a consequence, we find a new Hodge integral identity: if  $g \geq 2$ , then

$$(2) \quad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}],$$

and also a vanishing result: if  $g \geq 2$ , then for any  $1 \leq t \leq g-1$ , we have

$$(3) \quad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2t}(\mathbb{E}) \psi_1^{2(g-1-t)} = 0.$$

Recently, Liu-Xu [8] derived a generalized formula for Hodge integrals of type (2) by using the  $\lambda_g$  conjecture.

The rest of this paper is organized as follows: In Section 2, we recall the Mariño-Vafa formula and the Mumford's relations. In Section 3, we prove our main theorem and derive a new Hodge integral identity. In Section 4, we give another simple proof

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of  $\lambda_g$  conjecture. In Section 5, we derive a recursion formula of  $\lambda_{g-1}$ -integrals. In the last section, we list some low genus examples.

## 2. Preliminaries

**2.1. Partitions.** A partition  $\mu$  of a positive integer  $d$  is a sequence of integers  $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0$  such that

$$\mu_1 + \cdots + \mu_{l(\mu)} = d = |\mu|,$$

for each positive integer  $i$ , let

$$m_i(\mu) = |\{j | \mu_j = i, 1 \leq j \leq l(\mu)\}|.$$

The automorphism group  $\text{Aut}(\mu)$  of  $\mu$  consists of possible permutations among the  $\mu_i$ 's, hence its order is given by

$$|\text{Aut}(\mu)| = \prod_i m_i(\mu)!,$$

define the numbers

$$\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1), \quad z_\mu = \prod_j m_j(\mu)! j^{m_j(\mu)}.$$

The Young diagram of  $\mu$  has  $l(\mu)$  rows of adjacent squares: the  $i$ -th row has  $\mu_i$  squares. The diagram of  $\mu$  can be defines as the set of points  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$  such that  $1 \leq j \leq \mu_i$ , the conjugate of a partition  $\mu$  is the partition  $\mu'$  whose diagram is the transpose of the diagram  $\mu$ . Finally, we introduce the hook length of  $\mu$  at the squire  $x \in (i, j)$ :

$$h(x) = \mu_i + \mu'_j - i - j + 1.$$

Each partition  $\mu$  of  $d$  corresponds to a conjugacy class  $C(\mu)$  of the symmetric group  $S_d$  and each partition  $\nu$  corresponds to an irreducible representation  $R_\nu$  of  $S_d$ , let  $\chi_\nu(C(\mu)) = \chi_{R_\nu}(C(\mu))$  be the value of the character  $\chi_{R_\nu}$  on the conjugacy class  $C(\mu)$ .

**2.2. Mariño-Vafa formula.** Let  $\overline{\mathcal{M}}_{g,n}$  denote the Deligne-Mumford moduli stack of stable curves of genus  $g$  with  $n$  marked points. Let  $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  be the universal curve, and let  $\omega_\pi$  be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank  $g$  vector bundle over  $\overline{\mathcal{M}}_{g,n}$  whose fiber over  $[(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$  is  $H^0(C, \omega_C)$ , the complex vector space of holomorphic one forms on  $C$ . Let  $s_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$  denote the section of  $\pi$  which corresponds to the  $i$ -th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over  $\overline{\mathcal{M}}_{g,n}$  whose fiber over  $[(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$  is the cotangent line  $T_{x_i}^* C$  at the  $i$ -th marked point  $x_i$ . Consider the Hodge integral

$$(4) \quad \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

where  $\psi_i = c_1(\mathbb{L}_i)$  is the first Chern class of  $\mathbb{L}_i$ , and  $\lambda_j = c_j(\mathbb{E})$  is the  $j$ -th Chern class of  $\mathbb{E}$ . The dimension of  $\overline{\mathcal{M}}_{g,n}$  is  $3g - 3 + n$ , hence (4) is equal to zero unless  $\sum_{i=1}^n j_i + \sum_{i=1}^g ik_i = 3g - 3 + n$ . Let

$$(5) \quad \Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g = \sum_{i=0}^g (-1)^i \lambda_i u^{g-i}$$

be the Chern polynomial of the dual bundle  $\mathbb{E}^\vee$  of  $\mathbb{E}$ . For any partition  $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0$ , define

$$(6) \quad \mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\text{Aut}(\mu)|} [\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \\ \cdot \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau-1) \Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)},$$

$$(7) \quad \mathcal{C}_\mu(\lambda; \tau) = \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g,\mu}(\tau),$$

here  $\tau$  is a formal variable. Note that

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau-1) \Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} = |\mu|^{l(\mu)-3}$$

for  $l(\mu) \geq 3$ , and we use this expression to extend the definition to the case  $l(\mu) < 3$ .

Introduce formal variables  $p = (p_1, p_2, \dots, p_n, \dots)$ , and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for a partition  $\mu$ . Define generating functions

$$(8) \quad \mathcal{C}(\lambda; \tau, p) = \sum_{|\mu| \geq 1} \mathcal{C}_\mu(\lambda; \tau) p_\mu,$$

$$(9) \quad \mathcal{C}(\lambda; \tau, p)^\bullet = e^{\mathcal{C}(\lambda; \tau, p)}.$$

In [6], Chiu-chu Melissa Liu, Kefeng Liu and Jian Zhou have proved the following formula which was conjectured by Mariño and Vafa in [10].

**Theorem 2.1. (Mariño-Vafa Formula)** *For every partition  $\mu$ , we have*

$$\mathcal{C}(\lambda; \tau, p) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\mu} \left( \sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} V_{\nu^i}(\lambda) \right) p_\mu, \\ \mathcal{C}(\lambda; \tau, p)^\bullet = \sum_{|\mu| \geq 0} \left( \sum_{|\nu|=|\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2} V_\nu(\lambda) \right) p_\mu,$$

where

$$V_\nu(\lambda) = \prod_{1 \leq a < b \leq l(\nu)} \frac{\sin[(\nu_a - \nu_b + b - a)\lambda/2]}{\sin[(b - a)\lambda/2]} \frac{1}{\prod_{i=1}^{l(\nu)} \prod_{v=1}^{\nu_i} 2\sin[(v - i + l(\nu))\lambda/2]}.$$

It is known that

$$V_\nu(\lambda) = \frac{1}{2^{l(\nu)} \prod_{x \in \nu} \sin[h(x)\lambda/2]}.$$

**2.3. Mumford's relations.** Let  $c_t(\mathbb{E}) = \sum_{i=0}^g t^i \lambda_i$ , then we have

$$c_{-t}(\mathbb{E}) = t^g \Lambda_g^\vee \left( \frac{1}{t} \right).$$

Mumford's relations [11] are given by

$$(10) \quad c_t(\mathbb{E}) c_{-t}(\mathbb{E}) = 1,$$

equivalently

$$(11) \quad \Lambda_g^\vee(t) \Lambda_g^\vee(-t) = (-1)^g t^{2g},$$

then

$$(12) \quad \lambda_k^2 = \sum_{i=1}^k (-1)^{i+1} 2 \lambda_{k-i} \lambda_{k+i},$$

where  $\lambda_0 = 1$  and  $\lambda_k = 0$  for  $k > g$ . Let  $x_1, \dots, x_g$  be the formal Chern roots of  $\mathbb{E}$ , the Chern character is defined by

$$\text{ch}(\mathbb{E}) = \sum_{i=1}^g e^{x_i} = g + \sum_{n=1}^{+\infty} \sum_{i=1}^g \frac{x_i^n}{n!},$$

we write

$$(13) \quad \text{ch}_0(\mathbb{E}) = g,$$

$$(14) \quad \text{ch}_n(\mathbb{E}) = \frac{1}{n!} \sum_{i=1}^g x_i^n, \quad n = 1, 2, \dots.$$

From the above identities we have the relation between  $\text{ch}_n(\mathbb{E})$  and  $\lambda_n$ :

$$(15) \quad n! \text{ch}_n(\mathbb{E}) = \sum_{i+j=n} (-1)^{i-1} i \lambda_i \lambda_j, \quad n < 2g,$$

$$(16) \quad \text{ch}_n(\mathbb{E}) = 0, \quad n \geq 2g.$$

It is easy to see that

$$\begin{aligned} (2g-1)! \text{ch}_{2g-1}(\mathbb{E}) &= (-1)^{g-1} \lambda_{g-1} \lambda_g, \\ (2g-2)! \text{ch}_{2g-2}(\mathbb{E}) &= (-1)^{g-1} ((2g-2) \lambda_{g-2} \lambda_g - (g-1) \lambda_{g-1}^2), \\ (2g-3)! \text{ch}_{2g-3}(\mathbb{E}) &= (-1)^{g-1} (3 \lambda_{g-3} \lambda_g - \lambda_{g-1} \lambda_{g-2}). \end{aligned}$$

**2.4. Bernoulli numbers.** The Bernoulli numbers  $B_m$  are defined by the following series expansion:

$$(17) \quad \frac{t}{e^t - 1} = \sum_{m=0}^{+\infty} B_m \frac{t^m}{m!},$$

the first few terms are given by

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}.$$

Finally we recall two formulas which will be used later:

$$(18) \quad \frac{t/2}{\sin(t/2)} = 1 + \sum_{g=1}^{+\infty} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} t^{2g},$$

$$(19) \quad \sum_{i=1}^{d-1} i^m = \sum_{k=0}^m \frac{\binom{m+1}{k}}{m+1} B_k d^{m+1-k},$$

where  $m$  is a positive integer.

### 3. Some New Results from Mariño-Vafa Formula

In this section we derive some new results from the Mariño-Vafa formula, we will need two formulas in [7, 2.1 and 5.1].

**Theorem 3.1.** *We have the following results:*

$$(20) \quad \sum_{g \geq 0} \lambda^{2g} \int_{\mathcal{M}_{g,1}} \frac{\frac{d}{d\tau} \big|_{\tau=0} [\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)]}{1 - d\psi_1} \\ = - \sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{d\sin(d\lambda/2)} + \sum_{i+j=d, i, j \neq 0} \frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)},$$

$$(21) \quad \frac{d}{d\tau} \bigg|_{\tau=0} [\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)] = -\lambda_{g-1} - \lambda_g \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}).$$

**3.1. The coefficient of  $\lambda^{2g}$ .** Introduce the notation

$$b_g = \begin{cases} 1, & g = 0, \\ \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0, \end{cases}$$

then the coefficient of  $\lambda^{2g}$  in  $-\sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{d\sin(d\lambda/2)}$  is

$$(22) \quad \left( - \sum_{a=1}^{d-1} \frac{1}{a} \right) \cdot b_g d^{2g-1}.$$

If  $g_1, g_2 \geq 0$  and  $g_1 + g_2 = g$ , define

$$(23) \quad F_{g_1, g_2}(d) = \sum_{i+j=d, i, j \neq 0} i^{2g_1-1} j^{2g_2-1}.$$

In [6] it is showed that if  $g_1, g_2 \geq 1$ , then

$$(24) \quad F_{g_1, g_2}(d) = \sum_{k=0}^{2g_2-1} \sum_{l=0}^{2g-2-k} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{l} B_l d^{2g-1-l},$$

for the rest case we have

$$\begin{aligned}
 (25) \quad F_{0,g}(d) &= \sum_{i+j=d, i, j \neq 0} i^{-1} j^{2g-1} \\
 &= \sum_{i=1}^{d-1} i^{-1} (d-i)^{2g-1} \\
 &= \sum_{k=0}^{2g-1} (-1)^{2g-1-k} \binom{2g-1}{k} d^k \sum_{i=1}^{d-1} i^{2g-2-k} \\
 &= \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} d^k \sum_{l=0}^{2g-k-2} \frac{\binom{2g-1-k}{l}}{2g-1-k} B_l d^{2g-1-k-l} \\
 &\quad - (2g-1) d^{2g-2} (d-1) + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i} \\
 &= \sum_{k=0}^{2g-2} \sum_{l=0}^{2g-k-2} \binom{2g-1}{k} \binom{2g-1-k}{l} \frac{(-1)^{2g-1-k}}{2g-1-k} B_l d^{2g-1-l} \\
 &\quad + (2g-1) d^{2g-2} + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i}.
 \end{aligned}$$

Note that  $F_{0,g}(d) = F_{g,0}(d)$  and

$$(26) \quad \sum_{i+j=d, i, j \neq 0} \frac{\lambda^2}{8 \sin(i\lambda/2) \sin(j\lambda/2)} = \frac{1}{2} \sum_{g \geq 0} \lambda^{2g} \left( \sum_{g_1+g_2=g} b_{g_1} b_{g_2} F_{g_1, g_2}(d) \right).$$

**3.2. The Main Theorem.** Let

$$\sum_{g \geq 0} \lambda^{2g} LHS = \sum_{g \geq 0} \lambda^{2g} \int_{\mathcal{M}_{g,1}} \frac{\frac{d}{d\tau} |_{\tau=0} [\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau-1)]}{1 - d\psi_1},$$

and

$$\sum_{g \geq 0} \lambda^{2g} RHS = - \sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{d \sin(d\lambda/2)} + \sum_{i+j=d, i, j \neq 0} \frac{\lambda^2}{8 \sin(i\lambda/2) \sin(j\lambda/2)},$$

then we have

$$\begin{aligned}
 (27) \quad LHS &= - \int_{\mathcal{M}_{g,1}} (\lambda_{g-1} \psi_1^{2g-1}) d^{2g-1} \\
 &\quad - \sum_{k=0}^{2g-2} \left[ (2g-2-k)! (-1)^{2g-3-k} \int_{\mathcal{M}_{g,1}} \lambda_g \text{ch}_{2g-2-k}(\mathbb{E}) \psi_1^k \right] d^k,
 \end{aligned}$$

$$(28) \quad RHS = - \sum_{a=1}^{d-1} \frac{b_g}{a} d^{2g-1} + b_g F_{0,g}(d) + \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} F_{g_1, g_2}(d).$$

Hence we can derive our main theorem:

**Theorem 3.2.** *If  $1 \leq m \leq 2g - 3$  and  $g \geq 2$ , then*

$$\begin{aligned} & - (2g - 2 - m)!(-1)^{2g-3-m} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \text{ch}_{2g-2-m}(\mathbb{E}) \psi_1^m \\ & = b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m} + \\ & \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, m-1)} \frac{(-1)^{2g_2-1-k}}{2g_2-1-k} \binom{2g_2-1}{k} \binom{2g_2-1-k}{2g_2-1-m} B_{2g_2-1-m}. \end{aligned}$$

**Remark 3.3.** Liu-Liu-Zhou [7] have only considered the cases  $m = 2g - 1$  and  $m = 1$ .

**3.3. The case of  $m = 2g - 3$ .** If  $m = 2g - 3$ , we find that  $1! \text{ch}_1(\mathbb{E}) = \lambda_1$ , then

$$\begin{aligned} LHS &= - \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \lambda_1 \psi_1^{2g-3}, \\ RHS &= b_g \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2} B_2 \\ &+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g_2-1-k} \binom{2g_2-1}{k} \binom{2g_2-1-k}{2} B_2. \end{aligned}$$

From the above formula we obtain a new result of the Hodge integral.

**Theorem 3.4.** *If  $g \geq 2$ , then*

$$(29) \quad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}].$$

*Proof.* Note that

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} &= -b_g B_2 \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2} \\ &- \frac{B_2}{2} \sum_{g_1+g_2=g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g_2-1-k} \binom{2g_2-1}{k} \binom{2g_2-1-k}{2}, \end{aligned}$$

let us write

$$\begin{aligned} A_1 &= \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2}, \\ f_1(x) &= \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} (2g-2-k) x^{2g-3-k}, \\ g_1(x) &= \sum_{k=0}^{2g-2} (-1)^{2g-1-k} \binom{2g-1}{k} x^{2g-2-k}, \end{aligned}$$

then

$$A_1 = \frac{1}{2} \sum_{k=0}^{2g-4} (-1)^{2g-1-k} \binom{2g-1}{k} (2g-2-k), \quad xg_1(x) = (1-x)^{2g-1} - 1, \quad f_1(x) = g'_1(x).$$

Hence

$$f_1(x) = \frac{(2g-1)x(1-x)^{2g-2} - (1-x)^{2g-1} + 1}{x^2}, \quad f_1(1) = 1,$$

and we obtain

$$A_1 = \frac{1}{2} \left[ f_1(1) - \binom{2g-1}{2g-3} \right] = -\frac{1}{2} \left[ \binom{2g-1}{2g-3} - 1 \right].$$

Similarly, we write

$$A_2 = \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2},$$

then

$$A_2 = \begin{cases} \frac{1}{2} \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} (2g-2-k) \binom{2g_2-1}{k}, & g_2 \leq g-2, \\ \frac{1}{2} \sum_{k=0}^{2g-4} (-1)^{2g-1-k} (2g-2-k) \binom{2g-3}{k}, & g_2 = g-1. \end{cases}$$

**Case 1:**  $g_2 \geq g-2$ . Let

$$\begin{aligned} f_2(x) &= \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} (2g-2-k) \binom{2g_2-1}{k} x^{2g-3-k}, \\ g_2(x) &= \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} \binom{2g_2-1}{k} x^{2g-2-k}. \end{aligned}$$

Since  $g \geq g_2 + 2$ , then  $2g-3-(2g_2-1) \geq 2 > 0$  and  $g'_2(x) = f_2(x)$ . On the other hand

$$g_2(x) = (1-x)^{2g_2-1} x^{2g_1-1},$$

hence

$$g'_2(x) = -(2g_2-1)(1-x)^{2g_2-2} x^{2g_1-1} + (2g_1-1)(1-x)^{2g_2-1} x^{2g_1-2}$$

and

$$f_2(1) = \begin{cases} -1, & g_2 = 1, \\ 0, & 1 < g_2 \leq g-2. \end{cases}$$

**Case 2:**  $g_2 = g-1$ . let

$$\begin{aligned} f_3(x) &= \sum_{k=0}^{2g-4} (-1)^{2g-1-k} (2g-2-k) \binom{2g-3}{k} x^{2g-3-k}, \\ g_3(x) &= \sum_{k=0}^{2g-4} (-1)^{2g-1-k} \binom{2g-3}{k} x^{2g-2-k}. \end{aligned}$$

Since  $2g-3-(2g-4) = 1 > 0$ ,

$$\frac{g_3(x)}{x} = \sum_{k=0}^{2g-4} (-1)^{2g-3-k} \binom{2g-3}{k} x^{2g-3-k} = (1-x)^{2g-3} - 1$$



and

$$g'_3(x) = (1-x)^{2g-3} - 1 - (2g-3)(1-x)^{2g-4}x,$$

therefore we have

$$f_3(1) = \begin{cases} -2, & g = 2, \\ -1, & g > 2. \end{cases}$$

From the values of  $f_1(1), f_2(1), f_3(1)$ , we obtain

$$\begin{aligned} & \int_{\mathcal{M}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} \\ &= -b_g B_2 A_1 - \frac{B_2}{2} \left[ \frac{1}{2} \sum_{g_1+g_2=g, 1 \leq g_2 \leq g-2} b_{g_1} b_{g_2} f_2(1) + \frac{1}{2} b_1 b_{g-1} f_3(1) \right] \\ &= \frac{B_2}{2} [-b_g A_1 + b_1 b_{g-1}] \\ &= \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}] \end{aligned}$$

□

Since  $B_n = 0$  for  $n$  odd and  $n > 1$ , we also have the following vanishing result.

**Theorem 3.5.** *If  $g \geq 2$ , then for any  $1 \leq t \leq g-1$ , we have*

$$(30) \quad \int_{\mathcal{M}_{g,1}} \lambda_g \text{ch}_{2t}(\mathbb{E}) \psi_1^{2(g-1-t)} = 0.$$

#### 4. Another Simple Proof of $\lambda_g$ Conjecture

Let  $|\mu| = d, l(\mu) = n$ , denote by  $[\mathcal{C}_{g,\mu}(\tau)]_k$  the coefficient of  $\tau^k$  in the polynomial  $\mathcal{C}_{g,\mu}(\tau)$ , and let

$$\begin{aligned} J_{g,\mu}^0(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)} \mathcal{C}_{g,\mu}(\tau), \\ J_{g,\mu}^1(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)-1} \left( \sum_{\nu \in J(\mu)} I_1(\nu) \mathcal{C}_{g,\nu}(\tau) + \sum_{\nu \in C(\mu)} I_2(\nu) \mathcal{C}_{g-1,\nu}(\tau) \right. \\ &\quad \left. + \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) \mathcal{C}_{g_1,\nu^1}(\tau) \mathcal{C}_{g_2,\nu^2}(\tau) \right). \end{aligned}$$

The set  $J(\mu)$  consists of partitions of  $d$  of the form

$$\nu = (\mu_1, \dots, \widehat{\mu}_i, \dots, \mu_{l(\mu)}, \mu_i + \mu_j)$$

and the set  $C(\mu)$  consists of partitions of  $d$  of the form

$$\nu = (\mu_1, \dots, \widehat{\mu}_i, \dots, \mu_{l(\mu)}, j, k)$$

where  $j+k = \mu_i$ . The definitions of  $I_1, I_2$  and  $I_3$  can be found in [5]. Liu-Liu-Zhou [6] have proved the following differential equation:

$$(31) \quad \frac{d}{d\tau} J_{g,\mu}^0(\tau) = -J_{g,\mu}^1(\tau).$$

It is straightforward to check that

$$\begin{aligned} [\mathcal{C}_{g,\mu}(\tau)]_{n-1} &= -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)}, \\ \left[ \sum_{\nu \in J(\mu)} I_1(\nu) \mathcal{C}_{g,\nu}(\tau) \right]_{n-2} &= -\frac{\sqrt{-1}^{d+n-1}}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1} (1 - \mu_i \psi_i)}, \\ \left[ \sum_{\nu \in C(\mu)} I_2(\nu) \mathcal{C}_{g-1,\nu}(\tau) \right]_{n-2} &= 0, \\ \left[ \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) \mathcal{C}_{g_1,\nu^1}(\tau) \mathcal{C}_{g_2,\nu^2}(\tau) \right]_{n-2} &= 0, \end{aligned}$$

hence, from (31) we have the identity

$$(32) \quad \frac{n-1}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} = \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)}.$$

**Theorem 4.1.** *For any partition  $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$  of  $d$  and  $g > 0$ , then*

$$(33) \quad \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} = d^{2g+n-3} b_g.$$

*Proof.* Recall the definition of  $I_1(\nu)$ , where  $\nu = (\mu_1, \cdots, \widehat{\mu}_i, \cdots, \widehat{\mu}_j, \cdots, \mu_n, \mu_i + \mu_j)$ :

$$I_1(\nu) = \frac{\mu_i + \mu_j}{1 + \delta_{\mu_j}^{\mu_i}} m_{\mu_i + \mu_j}(\nu),$$

and it is easy to see that

$$\frac{m_{\mu_i + \mu_j}(\nu)}{|\text{Aut}(\nu)|} = \frac{m_{\mu_i}(\mu)(m_{\mu_j}(\mu) - \delta_{\mu_j}^{\mu_i})}{|\text{Aut}(\mu)|}.$$

Let

$$\mu : \underbrace{\mu_{k_1} = \cdots = \mu_{k_1}}_{t_1} > \underbrace{\mu_{k_2} = \cdots = \mu_{k_2}}_{t_2} > \cdots > \underbrace{\mu_{k_s} = \cdots = \mu_{k_s}}_{t_s} > 0,$$

where

$$\sum_{i=1}^s t_i = n, \quad \sum_{i=1}^s t_i \mu_{k_i} = d,$$

then

$$\begin{aligned}
& \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \\
&= \frac{1}{|\text{Aut}(\mu)|} \sum_{\nu \in J(\mu)} \frac{\mu_i + \mu_j}{1 + \delta_{\mu_j}^{\mu_i}} [m_{\mu_i}(\mu)(m_{\mu_j}(\mu) - \delta_{\mu_j}^{\mu_i})] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[ \frac{1}{2} \sum_{i=1}^s \sum_{j \neq i} (\mu_{k_i} + \mu_{k_j}) m_{\mu_{k_i}}(\mu) m_{\mu_{k_j}}(\mu) + \sum_{i=1}^s \mu_{k_i} m_{\mu_{k_i}}(\mu) (m_{\mu_{k_i}}(\mu) - 1) \right] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[ \frac{1}{2} \sum_{i=1}^s \sum_{j \neq i} (\mu_{k_i} + \mu_{k_j}) t_i t_j + \sum_{i=1}^s \mu_{k_i} t_i (t_i - 1) \right] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[ \sum_{i=1}^s \sum_{j \neq i} \mu_{k_j} t_j t_i + \sum_{i=1}^s \mu_{k_i} t_i^2 - d \right] \\
&= \frac{1}{|\text{Aut}(\mu)|} \left[ \sum_{i=1}^s t_i (d - \mu_{k_i} t_i) + \sum_{i=1}^s \mu_{k_i} t_i^2 - d \right] \\
&= \frac{(n-1)d}{|\text{Aut}(\mu)|}.
\end{aligned}$$

By the induction of  $n$  and the initial value of the Mariño-Vafa formula

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_g}{1 - \mu_1 \psi_1} = d^{2g-2} b_g,$$

we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} = d \cdot d^{2g+n-1-3} b_g = d^{2g+n-3} b_g.$$

□

**Corollary 4.2.** *The following  $\lambda_g$  conjecture is true:*

$$(34) \quad \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} = \binom{2g+n-3}{k_1, \dots, k_n} b_g,$$

where  $g > 0$ .

## 5. A recursion Formula of the $\lambda_{g-1}$ Integral

E.Getzler, A.Okounkov and R.Pandharipande have derived explicit formula for the multipoint series of  $\mathbb{CP}^1$  in degree 0 from the Toda hierarchy [2], then they obtained certain formulas for the Hodge integrals  $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{g-1} \psi_1^{k_1} \cdots \psi_n^{k_n}$ . In this section we give an effective recursion formula of the  $\lambda_{g-1}$  integrals using Mariño-Vafa formula. It is straightforward to check the following lemma.

**Lemma 5.1.** *We have the following identities*

$$\begin{aligned} \left[ \prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \right]_0 &= 1, \\ \left[ \prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \right]_1 &= \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}, \\ [\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau - 1) \Lambda_g^\vee(\tau)]_0 &= \lambda_g, \\ [\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau - 1) \Lambda_g^\vee(\tau)]_0 &= -\lambda_{g-1} - \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g, \end{aligned}$$

and

$$\begin{aligned} [\mathcal{C}_{g,\mu}(\tau)]_{n-1} &= -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)}, \\ [\mathcal{C}_{g,\mu}(\tau)]_n &= -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \left[ n - 1 + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right] \int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\ &\quad + \frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)}. \end{aligned}$$

Now, we can state our main theorem in this section using equation (31).

**Theorem 5.2.** *For any partition  $\mu$  with  $l(\mu) = n$ , we have the following recursion formula*

$$\begin{aligned} &\frac{n}{|\text{Aut}(\mu)|} \left[ n - 1 + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right] \int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\ &- \frac{n}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\ &= \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \left[ n - 2 + \sum_{i=1}^{n-1} \sum_{a=1}^{\nu_i-1} \frac{\nu_i}{a} \right] \int_{\mathcal{M}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)} \\ &- \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\mathcal{M}_{g,n-1}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k! (-1)^{k-1} \text{ch}_k(\mathbb{E}) \lambda_g}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)} \\ &+ \sum_{g_1+g_2=g, g_1, g_2 \geq 0} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} \frac{I_3(\nu^1, \nu^2)}{|\text{Aut}(\nu^1)| |\text{Aut}(\nu^2)|} \\ &\cdot \int_{\mathcal{M}_{g_1, n_1}} \frac{\lambda_{g_1}}{\prod_{i=1}^{n_1} (1 - \nu_i^1 \psi_i)} \int_{\mathcal{M}_{g_2, n_2}} \frac{\lambda_{g_2}}{\prod_{i=1}^{n_2} (1 - \nu_i^2 \psi_i)}. \end{aligned}$$

**5.1. The  $\lambda_g$ -Integral.** In this subsection, we re-derive the  $\lambda_g$ -integral from theorem 5.2.. Let  $\mu_i = Nx_i$  for some  $N \in \mathbb{N}$  and  $x_i \in \mathbb{R}$ , from Kim-Liu[4]'s method and

consider the coefficients of  $\ln N N^{2g+n-2}$  in theorem 5.2., then

$$\begin{aligned} & n(x_1 + \cdots + x_n) \prod_{l=1}^n x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (x_i + x_j)^{k_i+k_j} (x_1 + \cdots + x_n) \prod_{l \neq i,j} x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l} \\ &+ (x_1 + \cdots + x_n) \prod_{l=1}^n x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l}, \end{aligned}$$

i.e.

$$\begin{aligned} (35) \quad & (n-1) \prod_{l=1}^n x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (x_i + x_j)^{k_i+k_j} \prod_{l \neq i,j} x_l^{k_l} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l}. \end{aligned}$$

After introducing the formal variables  $s_l \in \mathbb{R}^+$  and applying the Laplace transformation

$$\int_0^{+\infty} x^k e^{-x/2s} dx = k!(2s)^{k+1}, \quad s > 0,$$

we select the coefficient of  $\prod_{l=1}^n (2s_l)^{k_l+1}$  from the transformation of (35), then we derive

$$(36) \quad (n-1) \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} = \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \frac{(k_i + k_j)!}{k_i! k_j!} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l}.$$

By the induction of  $n$ , we obtain the  $\lambda_g$  conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^n \psi_l^{k_l} = \binom{2g+n-3}{k_1, \dots, k_n} b_g,$$

in fact, in (36) we have

$$\begin{aligned} RHS &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \frac{(k_i + k_j)!}{k_i! k_j!} \frac{(2g+n-4)!}{\prod_{l \neq i,j} k_l! (k_i + k_j - 1)!} b_g \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \frac{k_i + k_j}{2g+n-3} \binom{2g+n-3}{k_1, \dots, k_n} b_g, \end{aligned}$$

note that  $k_1 + \cdots + k_n = 2g+n-3$ , therefore

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (k_i + k_j) &= \frac{1}{2} \sum_{i=1}^n [(n-1)k_i + (2g+n-3-k_i)] \\ &= \frac{1}{2} [(n-2)(2g+n-3) + (2g+n-3)n] \\ &= \frac{1}{2} [(2n-2)(2g+n-3)] \\ &= (n-1)(2g+n-3). \end{aligned}$$

**5.2. The Recursion Formula of  $\lambda_{g-1}$ -integral.** We have found the *singular part*  $\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}$  in theorem 5.2., using the following theorem, we can eliminate this part and derive the recursion formula of  $\lambda_{g-1}$ -integral. The notation  $[F]_{sing}$  means the singular part of  $F$ . First, in theorem 5.2., we have

$$\begin{aligned} \left[ \frac{LHS}{d^{2g+n-4}b_g} \right]_{sing} &= n \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d, \\ \left[ \frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \left[ \sum_{l \neq i, j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} + (\mu_i + \mu_j) \sum_{a=1}^{\mu_i+\mu_j-1} \frac{1}{a} \right] \\ &\quad + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a}. \end{aligned}$$

**Theorem 5.3.** *Under the above notation, we have*

$$\left[ \frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} = \left[ \frac{LHS}{d^{2g+n-4}b_g} \right]_{sing} + 2(n-1)d.$$

*Proof.* Since

$$\begin{aligned} &\sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a} \\ &= \sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_i(\mu_i + \mu_j)}{a} \\ &= \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j} \frac{(\mu_i + \mu_j)^2}{a} \\ &\quad - \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{\mu_i + \mu_j}{a} + \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j} \frac{(\mu_i + \mu_j)}{a} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} \\ &\quad + \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)}{a} - \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j), \end{aligned}$$

where we use the identity

$$\begin{aligned} \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)\mu_i}{a} &= \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)\mu_j}{a} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a}. \end{aligned}$$

Note that  $\sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) = 2(n-1)d$ , hence

$$\begin{aligned}
& \left[ \frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} + \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d \\
&+ \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^n \sum_{j \neq i} \mu_i \sum_{a=1}^{\mu_j-1} \frac{(\mu_i + \mu_j)}{a} + \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \\
&= \left( \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} \\
&+ \sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a} + 2(n-1)d,
\end{aligned}$$

it is straightforward to check that

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \sum_{a=1}^{\mu_l-1} \frac{\mu_l}{a} &= (n-2) \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a}, \\
\sum_{i=1}^n \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j(\mu_i + \mu_j)}{a} &= \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} + \sum_{i=1}^n \sum_{j \neq i} \mu_j \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a}, \\
\sum_{i=1}^n \sum_{j \neq i} \mu_j \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} &= (n-1) \sum_{i=1}^n \mu_i \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
& \left[ \frac{RHS}{d^{2g+n-4}b_g} \right]_{sing} \\
&= \left( \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + (n-2) \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} \\
&+ \sum_{i=1}^n \mu_i \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} + (n-1) \sum_{i=1}^n \mu_i \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} + 2(n-1)d \\
&= \left( \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + (n-1) \sum_{i=1}^n \mu_i \left( \sum_{j \neq i} \sum_{a=1}^{\mu_j-1} \frac{\mu_j}{a} + \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) + 2(n-1)d \\
&= \left( \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + (n-1) \left( \sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) d + 2(n-1)d \\
&= \left[ \frac{LHS}{d^{2g+n-4}b_g} \right]_{sing} + 2(n-1)d.
\end{aligned}$$

□

Let  $\mathbb{R}^k[\mu_1, \dots, \mu_n]$  be the space of all homogeneous polynomials with real coefficients in  $\mu_1, \dots, \mu_n$  of degree  $k$ , then it is the subring of  $\mathbb{R}[\mu_1, \dots, \mu_n]$ . From the Theorem 5.3, we obtain the recursion formula of  $\lambda_{g-1}$  Hodge integral.

**Theorem 5.4.** *For any partition  $\mu$  with  $l(\mu) = n$  and  $|\mu| = d$ , we have the recursion formula*

$$\begin{aligned} & \frac{n}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_{g-1}}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \\ = & \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_{g-1}}{\prod_{i=1}^{n-1} (1 - \nu_i \psi_i)} \\ & - \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} \frac{I_3(\nu^1, \nu^2)}{|\text{Aut}(\nu^1)| |\text{Aut}(\nu^2)|} d_1^{2g_1+n_1-3} d_2^{2g_2+n_2-3} b_{g_1} b_{g_2}. \end{aligned}$$

under the ring  $\mathbb{R}^{2g-2+n}[\mu_1, \dots, \mu_n]$ , where  $l(\nu^i) = n_i$  and  $|\nu^i| = d_i$  for  $i = 1, 2$ .

**Remark 5.5.** When we consider the simplest case  $n = 1$ , the above identity become the formula used in [6].

## 6. Some Examples of The Main Theorem

In this section we give some examples of theorem 3.2.

**6.1. The case of  $g = 3$ .** If  $g = 3$ , then  $1 \leq m \leq 3$ . We consider three cases.

**6.1.1.  $m=1$ .**  $LHS = -3 \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \text{ch}_3(\mathbb{E}) \psi_1$ , and  $3\text{ch}_3(\mathbb{E}) = \sum_{i+j=3} (-1)^{i-1} i \lambda_i \lambda_j = 3\lambda_3 - \lambda_1 \lambda_2$ , then we get

$$\begin{aligned} LHS &= \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 (\lambda_1 \lambda_2 - 3\lambda_3) \psi_1 = \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_2 \lambda_3 \psi_1, \\ RHS &= b_3 \frac{(-1)^5}{5} \binom{5}{0} \binom{5}{4} B_4 + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \frac{-1}{5} \binom{2g_2-1}{0} \binom{5}{4} B_4 \\ &= -B_4 (b_3 + b_1 b_2). \end{aligned}$$

Since  $b_1 = \frac{1}{24}$ ,  $b_2 = \frac{7}{5760}$ ,  $b_3 = \frac{31}{967680}$ , we have

$$(37) \quad \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_2 \lambda_3 \psi_1 = \frac{1}{362880}.$$

**6.1.2.  $m=2$ .** In this case we have  $LHS = 2 \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \text{ch}_2(\mathbb{E}) \psi_1^2$ , and  $2! \text{ch}_2(\mathbb{E}) = 2\lambda_2 - \lambda_1^2$ ,  $B_3 = 0$ . Then we have

$$\begin{aligned} LHS &= \int_{\overline{\mathcal{M}}_{3,1}} (2\lambda_2 \lambda_3 - \lambda_3 \lambda_1^2) \psi_1^2, \\ RHS &= b_3 \sum_{k=0}^1 \frac{(-1)^{4-k}}{5-k} \binom{5}{k} \binom{5-k}{3} B_3 \\ &\quad + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^1 \frac{(-1)^{5-k}}{5-k} \binom{2g_2-1}{k} \binom{5-k}{3} B_3 \\ &= 0, \end{aligned}$$



hence

$$\int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2 = 2 \int_{\overline{\mathcal{M}}_{3,1}} \lambda_2 \lambda_3 \psi_1^2.$$

Using the formula  $\int_{\overline{\mathcal{M}}_{3,1}} \lambda_2 \lambda_3 \psi_1^2 = \frac{1}{120960}$ , we get

$$(38) \quad \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2 = \frac{1}{60480}.$$

**6.1.3.**  $m=3$ . In this case  $LHS = - \int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \text{ch}_1(\mathbb{E}) \psi_1^3$  and  $\text{ch}_1(\mathbb{E}) = \lambda_1$ , hence

$$\begin{aligned} LHS &= - \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_3 \psi_1^3, \\ RHS &= b_3 \sum_{k=0}^2 \frac{(-1)^{5-k}}{5-k} \binom{5}{k} \binom{5-k}{2} B_2 \\ &\quad + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2)} \frac{(-1)^{2g_2-1-k}}{5-k} \binom{2g_2-1}{k} \binom{5-k}{2} B_2 \\ &= -\frac{9}{2} b_3 B_2 - \frac{1}{2} b_1 b_2 B_2 \\ &= -\frac{41}{1451520}, \end{aligned}$$

so

$$(39) \quad \int_{\overline{\mathcal{M}}_{3,1}} \lambda_1 \lambda_3 \psi_1^3 = \frac{41}{145120}.$$

**Remark 6.1.** The values of (37) and (39) match with the results in [9], the identity (38) is a new result.

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## References

- [1] C.Faber, R. Pandharipande. *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173-199.
- [2] ———. *Hodge integrals, partition matrices, and the  $\lambda_g$  conjecture*, Ann. Math. (2) **157** (2003), 97-124. E. Getzler, A. Okounkov, R. Pandharipande. *Multipoint series of Gromov-Witten invariants of  $\mathbb{CP}^1$* , preprint: math.AG/0207106.
- [3] E. Getzler, A. Okounkov, R. Pandharipande. *Multipoint series of Gromov-Witten invariants of  $\mathbb{CP}^1$* , preprint: math.AG/0207106.
- [4] Y.-S. Kim, K. Liu. *A simple proof of Witten conjecture through localization*, preprint: math.AG/0508384.
- [5] A.M. Li, G. Zhao, Q. Zheng. *The number of ramified coverings of a Riemann surface by Riemann surface*, Comm.Math.Phys. **213** (2000), 685-696.

- [6] C.-C. Liu, K. Liu, J. Zhou. *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), 289-340.
- [7] C.-C. Liu, K. Liu, J. Zhou. *Mariño-Vafa formula and Hodge integral identities*, J. Algebraic Geom. **15** (2006), 379-398.
- [8] K. Liu, H. Xu. *Estimate of denominators of Hodge integrals*, preprint, 2006.
- [9] W. Lu. *A note of the Mariño-Vafa formula*, Science in China Ser. A Math. (11) **35** (2005), 1276-1287. (in chinese)
- [10] M. Mariño, C. Vafa. *Framed knots at large  $N$* , Orbifolds in mathematics and Physics (Madison, WI, 2001), 185-204, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.
- [11] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and geometry, Vol. II, Progr. Math., **36**, Birkhuser Boston, Boston, 1983, pp. 271-328.
- [12] J. Zhou. *Some closed formulas and conjectures for Hodge integrals*, Math. Res. Lett. **10** (2003), 275-286.
- [13] ———. *Hodge integrals, Hurwitz numbers, and symmetric group*, preprint: math.AG/0308024.

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